Study of Distributions Having Polynomial Hazard Function

By
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Thesis Defence Committee

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December, 1991

To My Grandfather

Haza'a

And My Grandmother

Khadra

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Abdel-Razzaq Mugdadi

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ABSTRACT

Bain (1974) studied the distributions having linear hazard function $h(t) = a + bt$. He obtained the least squares type estimates and also the maximum likelihood estimates of the parameters a and b.

In this thesis we generalize the work of Bain. First we study distributions having polynomial hazard function and later we obtain the least squares type estimates and also the maximum likelihood estimates of the different parameters. In most of the cases studied no analytic formula could be obtained for the mean and the variance of these estimates. Therefore, we did simulation investigations to study the bias and the mean squares error of these estimates.

In particular, we study the distributions having power hazard function $h(t) = at^k$.

دراسة توزيعات اقتران الخطورة كثير الحدود

عبد الرزاق احمد هزاع مقدادي

الملخص

في سنة ١٩٧٤ قام "بين" (Bain) بدراسة اقتران الخطورة الخطي ذي الشكل

$$h(t) = a + bt$$

حيث اوجد تقديرات من نوع اصغر المربعات (Least Square type Estimates) والتقديرات بطريقة الاحتمالات العظمى (Maximum Likelihood Estimates) للمعلمات a و b .

في هذه الرسالة قمنا بتعميم العمل الذي قام به بين . في البداية درسنا توزيع اقتران الخطورة كثير الحدود وقد اوجدنا تقديرات من نوع اصغر المربعات وتقديرات بطريقة الاحتمالات العظمى لمعلمات مختلفة .

في معظم الحالات التي درسناها لا نستطيع ايجاد الوسط والتباين لهذه التقديرات . لذلك قمنا ببحث بواسطة المحاكاة لدراسة متوسط الانحراف ومتوسط مربعات الاخطاء التي من خلالها نستطيع معرفة الوسط والتباين .

بشكل خاص قمنا بدراسة توزيع اقتران الخطورة الاسي

$$h(t) = at^{k}$$

CONTENTS

CHAPTER ONE

Introduction

In this chapter we introduce the concept of the reliability function of a unit and also the associated hazard function. Later we discuss some distribution functions and the associated hazard functions.

1.1 Reliability and the Hazard Function

The application of statistical reasoning to certain engineering problems gave birth in 1940 to the concept of reliability. The reliability of any unit at time t (under given conditions) is the probability that the unit has been working without failure in the interval of time $[0,t)$.

Assume that the distribution function of the life T of a unit at time t is $F(t)$. It is clear that the life of a unit is a non-negative number and therefore $F(t) = 0$ for $t \leq 0$. It follows that the corresponding derived function viz, the density function $f(t)$ and the hazard function $h(t)$ (to be defined later) will also be zero for $t \leq 0$. Therefore, throughout the thesis, we shall write the different functions for $t > 0$ only.

From the definition given earlier, it follows that the reliability function $R(t)$ of the unit at time t is given by

$$R(t) = P(T > t)$$

$$= 1 - F(t) \qquad t > 0 \qquad (1.1.1)$$

Assume that the distribution function F(t) is differentiable with respect to t (a condition that is satisfied in most applications). We define a new function

$$h(t) = \frac{\frac{d}{dt} F(t)}{1 - F(t)}$$

$$= - \frac{\frac{d}{dt} R(t)}{R(t)} \qquad t > 0 \qquad (1.1.2)$$

The function h(t) is called the hazard function of the unit at time t. It gives the instantaneous rate of failure of the unit at time t knowing that is was working in the interval [0,t). Furthermore, from (1.1.1) and (1.1.2) it follws that

$$F(t) = 1 - \exp\left[- \int_0^t h(x)\, dx \right] \qquad (1.1.3)$$

1.2 Some Distribution Functions & Associated Hazard Function

a. Exponential Distribution

The wellknown and most used hazard function is the constant function

$$h(t) = a \quad , a > 0, \ t > 0$$

Using (1.1.3) we obtain the corresponding distribution function as

$$F(t) = 1 - e^{-at} \quad , \quad a > 0,\ t > 0 \qquad (1.2.1)$$

This is the exponential distribution function. The corresponding density function f(t) is given by

$$f(t) = a\, e^{-at} \quad , \quad a > 0,\ t > 0 \qquad (1.2.2)$$

b. **Weibull Distribution**

The second most used distribution for the life of an item is known as Weibull distribution which is given by

$$F(t) = 1 - \exp\left[-\frac{t^p}{a} \right] \quad , \quad p > 0,\ a > 0,\ t > 0$$

Therefore, the corresponding hazard function is

$$h(t) = \frac{p}{a}\, t^{p-1} \quad , \quad p > 0,\ a > 0,\ t > 0$$

and the density function is

$$f(t) = \frac{p}{a}\, t^{p-1} \exp\left[-\frac{t^p}{a} \right] , \quad p > 0,\ a > 0,\ t > 0 \quad (1.2.2)$$

In (1.2.2) if we take P = 1 then we obtain the exponential distribution. Also, if we take P = 2 in (1.2.2) then we obtain

$$f(t) = \frac{2t}{a} \exp\left[-\frac{t^2}{a} \right] , \quad a > 0,\ t > 0$$

which is the density function of the Rayleigh distribution. Obviously, the hazard function of the Rayleigh distribution is given by

$$h(t) = \frac{2t}{a} \quad , \quad a > 0, \ t > 0$$

1.3 Literature

The literature on the hazard funtions and the corresponding life distribution functions is very large. For this purpose one can refer to the well known books on this subject and the bibliographies in these books. Some good references are given below.

1. Bain, L.J. (1978). "Statistical Analysis of Reliability and Life Testing Models". Marcel Deckker Inc., New York and Basel.

2. Kalbfleisch, J.D. and Prentice, R.L. (1980). "The Statistical Analysis of Failure Time Data". John Wiley and Sons, New York.

3. Mann, N.R, Schafer, R.E. and Singpurwalla, N.D. (1974). "Methods for Statistical Analysis of Reliability and Life Data". John Wiley and Sons, New York.

4. Sinha, S.K. (1986). "Reliability and Life Testing". John Wiley and Sons, New York.

The necessary part of the literature will be described in detail in the respective chapters of this thesis.

1.4 Summary

In Chapter Two we introduce the general polynomial hazard function. First we discuss the estimation of the parameters by least squares type method.

The linear hazard function $h(t) = a + bt$ has been discussed in detail by Bain (1974) and he also obtained the least squares type estimates of a and b. Later in the paper he obtained the maximum likelihood estimates of the parameters and also the asymptotic covariance matrix. He didnot discuss the bias of the estimates. In the absence of analytical formulae we investigate these properties using simulation for the maximum likelihood estimates of the parameters in Chapter Two and Chapter Three.

The power hazard function $h(t) = at^k$ is discussed in three cases: (i) a is the parameter and k is known, (ii) k is the parameter and a is known, and (iii) a and k are parameters. In Chapter Two we obtain the least squares type estimates for each of the above three cases. In Chapter Three we obtain the maximum likelihood estimates for each of the three cases.

In case (i) the mean and the variance of the estimates are well known. For the cases (ii) and (iii) we have no knowledge about the mean and the variance of the estimates. In order to obtain some knowledge about these properties we make simulation. From these simulations we obtain the mean bias and the mean squares error. Our conclusions will be given after the description of the respective simulations in Chapters Two and Three.

CHAPTER TWO

Least Squares Type Estimation

In this chapter we describe briefly in section (2.1) the general polynomial hazard function and obtain the estimates of the parameters by least squares type method using censored and complete samples. In case of the linear hazard function the results obtained by Bain (1974) are described in section (2.1). Finally, we find the least squares type estimates of the parameters in the power hazard function and the results of a simulation study are described in section (2.2).

2.1 Polynomial Hazard Function

Suppose that we have the k-th degree polynomial hazard function

$$h(t) = a_0 + a_1 t + \dots + a_k t^k \qquad (2.1.1)$$

where the parameters $a_0, a_1, \dots, a_k$ are real numbers.

Using equation (1.1.3) we have

$$F(t,\theta) = 1 - \exp\left[(-a_0 t + \frac{a_1 t^2}{2} + \dots + \frac{a_k t^{k+1}}{k+1})\right], \ t > 0 \quad (2.1.2)$$

where $\theta = (a_0, a_1, \dots, a_k)$.

Let $t_1, \dots, t_r$ denote the smallest r observations in a random sample of size n from the density function having the hazard function (2.1.1). Estimate of the vector parameter θ

may be obtained by minimizing over the parameter space the sum

$$\sum_{i=1}^{r} \left[\ln(1-F(t_i,\theta)) - \ln E(1-F(t_i,\theta))\right]^2 \qquad (2.1.3)$$

From (2.1.2) it follows that

$$-\ln[1-F(t_i,\theta)] = a_0 t_1 + \frac{a_1}{2} t_1^2 + \ldots + \frac{a_k}{k+1} t_1^{k+1}. \qquad (2.1.4)$$

and

$$\ln E(1-F(t_i,\theta)) = \ln(1-EF(t_i,\theta)) = \ln\left(1 - \frac{i}{n+1}\right), \; i = 1, \ldots, r$$

because $EF(t_i,\theta) = \dfrac{i}{n+1}$.

Using the notation $H(t_i,\theta) = -\ln[1-F(t_i,\theta)]$, and $\xi_i = -\ln(1-EF(t_i,\theta))$, $i = 1,\ldots,r$, (2.1.3) can be written as

$$\sum_{i=1}^{r} \left[H(t_i,\theta) - \xi_i \right]^2 \qquad (2.1.5)$$

It is clear that (2.1.5) will be minimized if we take $H(t_i,\theta) = \xi_i$, $i = 1, \ldots, r$.

This leads to a system of linear equations which in matrix notation can be written as

$$\xi = X\,\theta$$

where

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{bmatrix}, \qquad X = \begin{bmatrix} t_1 & t_1^2 & \ldots & t_1^{k+1} \\ t_2 & t_2 & \ldots & t_2 \\ \vdots & \vdots & & \vdots \\ t_r & t_r^2 & \ldots & t_r^{k+1} \end{bmatrix}$$

and

$$\theta = \begin{bmatrix} a_0 \\ a_{1/2} \\ \vdots \\ a_{k/(k+1)} \end{bmatrix}$$

If $(X'X)^{-1}$ exists then the estimate $\tilde{\theta}$ of θ is given by

$$\tilde{\theta} = (X'X)^{-1} X' \xi \qquad (2.1.6)$$

The estimate $\tilde{\theta}$ obtained is called a least squares type of θ.

A very important special case of the polynomial hazard function is the linear hazard function given by $h(t)=a_0+a_1t$. As mentioned earlier, this case has been studied by Bain (1974). Using the previous notation his results can be described as

$$\tilde{a}_0 = \frac{\sum_{i=1}^{r} t_i^4 \cdot \sum_{i=1}^{r} t_i \, \xi_i - \sum_{i=1}^{r} t_i^3 \cdot \sum_{i=1}^{r} t_i^2 \, \xi_i}{\sum_{i=1}^{r} t_i^2 \cdot \sum_{i=1}^{r} t_i^4 \, \xi_i - \left[\sum_{i=1}^{r} t_i^3 \right]^2}$$

and

$$\tilde{a}_1 = \frac{2 \left[\sum_{i=1}^{r} t_i^2 \cdot \sum_{i=1}^{r} t_i^2 \, \xi_i - \sum_{i=1}^{r} t_i^3 \cdot \sum_{i=1}^{r} t_i \, \xi_i \right]}{\sum_{i=1}^{r} t_i^2 \cdot \sum_{i=1}^{r} t_i^4 - \left[\sum_{i=1}^{r} t_i^3 \right]^2}$$

2.2 Power Hazard Function

In this section we obtain the least squares type estimates of the parameters in the distribution having the power hazard function

$$h(t) = a\,t^k \quad , \quad a > 0,\ k > -1,\ t > 0 \qquad (2.2.1)$$

Corresponding to this hazard function we have

$$F(t) = 1 - \exp\left[-\frac{a}{k+1}\,t^{k+1}\right]$$

and therefore the density function is

$$f(t) = a\,t^k \exp\left[-\frac{a}{k+1}\,t^{k+1}\right], \quad a>0,\ k>-1,\ t>0 \qquad (2.2.2)$$

For the hazard function $h(t) = a\,t^k$ it is well known that many life distributions are special cases of the distribution defined in (2.2.2).

It is easily seen that

$$E\,t^r = \left[\frac{k+1}{a}\right]^{\frac{r}{k+1}} \Gamma\left[1 + \frac{r}{k+1}\right]$$

exists if $k > -1$

In particular, it follows that

$$E\,t = \left[\frac{k+1}{a}\right]^{\frac{1}{k+1}} \Gamma\left[\frac{k+2}{k+1}\right]$$

$$E\,t^2 = \left[\frac{k+1}{a}\right]^{\frac{2}{k+1}} \Gamma\left[\frac{k+3}{k+1}\right]$$

and

$$Var(t) = \left(\frac{k+1}{a}\right)^{\frac{2}{k+1}} \left[\Gamma\left(\frac{k+3}{k+1}\right) - \left[\Gamma\left(\frac{k+2}{k+1}\right)\right]^2\right]$$

Therefore, we can conclude that for the existence of the moments (and in particular for the existence of the mean and the variance), it is necessary that $k > -1$.

Also, the mode t_m of the distribution exists and is given by

$$t_m = \left[\frac{k}{a} \right]^{\frac{1}{k+1}} \quad \text{if} \quad a > 0,\ k \neq 0,\ k > -1.$$

But for $k = 0$ we have

$$f(t) = a\, e^{-at} \quad , \qquad a > 0,\ t \geq 0.$$

It is easily seen that the mode exists in this case and is equal to zero.

In this section we obtain the least squares type estimates of the parameters for the distribution having a power hazard function $h(t) = at^k$ using the smallest r censored sample. The results for the complete sample are easily obtained by taking $r = n$.

2.2.1 a is the Parameter and k is Known

Suppose $t_1, \ldots, t_r$ denotes the smallest r observations of a sample of size n from the distribution having the power hazard function. Using equation (1.1.3) we obtain

$$F(t,\theta) = 1 - \exp\left[-\frac{a}{k+1}\, t^{k+1} \right] \qquad (2.2.3)$$

By using the previous notation the estimate of the parameter a may be obtained by minimizing over the parameter space the sum

$$\sum_{i=1}^{r}\left[\ln\left[1-F(t_i,\theta)\right]-\ln\left[1-EF(t_i,\theta)\right]\right]^2=\sum_{i=1}^{r}\left[H(t_i,\theta)-\xi_i\right]^2 \quad (2.2.4)$$

where $H(t,\theta) = -\ln\left[1 - F(t,\theta)\right]$

and

$$\xi_i = -\ln\left[1 - E\,F(t_i,\theta)\right]$$

It is clear that (2.2.4) will be minimized when $H(t_i) = \xi_i$, $i = 1, \ldots, r$. This leads to a system of linear equations which in matrix notation can be written as

$$\xi = X\,\theta$$

where

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_r \end{bmatrix}, \quad X = \begin{bmatrix} t_1^{k+1} \\ t_2^{k+1} \\ \vdots \\ t_r^{k+1} \end{bmatrix} \quad \text{and} \quad \theta = \frac{a}{k+1}$$

Then by (2.1.6) if $(X'X)^{-1}$ exists then

$$\tilde{\theta} = (X'X)^{-1}\,X'\,\xi$$

But

$$X'X = \sum_{i=1}^{r} t_i^{2k+2}$$

Therefore, $(X'X)^{-1}$ exists and is given by

$$(X'X)^{-1} = \frac{1}{\sum_{i=1}^{r} t_i^{2k+2}}$$

Also, $X'\xi = \sum_{i=1}^{r} t_i^{k+1}\,\xi_i$

Recall that

$$\xi_i = -\ln\left[1 - \frac{i}{n+1}\right], \qquad i = 1, \ldots, r$$

Therefore,

$$\tilde{\theta} = \frac{\tilde{a}}{k+1} = \frac{\sum_{i=1}^{r} t_i^{k+1}\,\xi_i}{\sum_{i=1}^{r} t_i^{2k+2}}$$

Finally, we have

$$\tilde{a} = (k+1)\frac{\sum_{i=1}^{r} t_i^{k+1}\,\xi_i}{\sum_{i=1}^{r} t_i^{2k+2}}$$

2.2.2 k is the Parameter and a is Known

We have from (2.1.5) that

$$L(k) = \sum_{i=1}^{r}\left(H(t_i,\theta) - \xi_i\right)^2$$

$$= \sum_{i=1}^{r}\left(\frac{a}{k+1}\,t_i^{k+1} - \xi_i\right)^2$$

Therefore, to obtain the values of k that minimize $L(k)$ we consider

$$\frac{\partial L(a,k)}{\partial k} = 2\sum_{i=1}^{r}\left[\frac{a}{k+1}\,t_i^{k+1} - \xi_i\right]\cdot\frac{a}{(k+1)^2}\left[t_i^{k+1}\ln t_i(k+1) - t_i^{k+1}\right]$$

To obtain the least squares type estimate of k, let

$\frac{\partial L(k)}{\partial k} = 0$. It follows that $\frac{\partial L(k)}{\partial k} = 0$ iff

$$\sum_{i=1}^{r} \ln t_i (t_i^{k+1}) \xi_i (k+1) - \sum_{i=1}^{r} t_i^{k+1} \xi_i = \frac{a}{k+1} \left[\sum_{i=1}^{r} \ln t_i t_i^{2k+2} (k+1) - \sum_{i=1}^{r} t_i^{2k+1} \right]. \quad (2.2.5)$$

Furthermore

$$\frac{\partial^2 L(k)}{\partial k^2} = \frac{2a^2}{(k+1)^4} \left[2(k+1)^2 D - 2(k+1) E \right] - \frac{2a^2}{(k+1)^6} \left[2E(k+1)^3 - 3(k+1)^2 F \right] - \frac{2a}{(k+1)^2} \left[A(k+1) - B \right] + \frac{2a}{(k+1)^4} \left[B(k+1)^2 - 2(k+1)C \right]$$

where

$$A = \sum_{i=1}^{r} \left(\ln t_i \right)^2 \left(t_i^{k+1} \right) \xi_i$$

$$B = \sum_{i=1}^{r} \left(\ln t_i \right) \left(t_i^{k+1} \right) \xi_i$$

$$C = \sum_{i=1}^{r} t_i^{k+1} \xi_i$$

$$D = \sum_{i=1}^{r} \left(\ln t_i \right)^2 t_i^{2k+2}$$

$$E = \sum_{i=1}^{r} \left(\ln t_i \right) \left(t_i^{2k+2} \right)$$

$$F = \sum_{i=1}^{r} t_i^{2k+2}$$

Therefore, $\dfrac{\partial^2 L(k)}{\partial k^2} > 0$

iff

$$a > \frac{(k+1)\left[A(k+1)^2 - 2\,[\, B(k+1) - C \,]\right]}{\left[2D(k+1)^2 - F - 4\,[\, E(k+1) - F \,] \right]}$$

Let $\tilde{k}$ be the value of k obtained for equation (2.4.6).

Therefore, $\left.\dfrac{\partial^2 L(k)}{\partial k^2}\right|_{k=\tilde{k}} > 0$

iff

$$a > \frac{(\tilde{k}+1)\left[\tilde{A}(\tilde{k}+1)^2 - 2\,[\, \tilde{B}(\tilde{k}+1) - \tilde{C} \,]\right]}{\left[2\tilde{D}(\tilde{k}+1)^2 - \tilde{F} - 4\,[\, \tilde{E}(\tilde{k}+1) - \tilde{F} \,] \right]} \qquad (2.2.6)$$

where

$$\tilde{A} = \sum_{i=1}^{r} \left(\ln t_i \right)^2 \left(t_i^{\tilde{k}+1} \right) \xi_i$$

$$\tilde{B} = \sum_{i=1}^{r} \left(\ln t_i \right) \left(t_i^{\tilde{k}+1} \right) \xi_i$$

$$\tilde{C} = \sum_{i=1}^{r} t_i^{\tilde{k}+1} \; \xi_i$$

$$\tilde{D} = \sum_{i=1}^{r} \left(\ln t_i \right)^2 \; t_i^{2\tilde{k}+2}$$

$$\tilde{E} = \sum_{i=1}^{r} \left(\ln t_i \right) \left(t_i^{2\tilde{k}+2} \right)$$

$$\tilde{F} = \sum_{i=1}^{r} t_i^{2\tilde{k}+2}$$

Therefore, when the condition (2.2.6) is satisfied then the least squares type estimate $\tilde{k}$ of k is the solution of k for equation (2.2.5).

Since we are unable to obtain an analytic expression for the mean and the variance of $\hat{k}$ we don't know whether the least square type estimate of the parameter k is unbiased or not and also we have no knowledge about it's variance. Therefore, we decided to study these quantities by simulation.

The first step to simulation is to obtain a random sample from the density function (2.2.2). For this purpose

1. Find the distribution function $F(t) = 1 - \exp\left[-\frac{a}{k+1}\, t^{k+1}\right]$

2. Set U = F(t) where U is a random number in [0,1).

   Therefore we have

$$U = 1 - \exp\left[-\frac{a}{k+1}\, t^{k+1}\right].$$

3. Find t defined by

$$t = \left[-\frac{k+1}{a}\,\ln(1 - U)\right]^{\frac{1}{k+1}} \qquad (2.2.7)$$

The following symbols will be used:

N : number of samples

a,k : variables

SS : sample size

MB : Mean Bias

MSE : Mean Squares Error

R : Number of failed units at which the experiment is terminated

The simulation procedure is as follows:

i. Fix the values of a, k, N, R and SS.

ii. Obtain a value of U and calcualte t using the relation (2.2.7).

iii. Repeat the process (ii) SS times. This gives us the sample $t_1, \ldots, t_{SS}$ and take the smallest R observations $t_1, \ldots, t_R$.

iv. Obtain the estimate $\tilde{k}$ of k using the equation (2.2.5) and then check the condition in inequality (2.2.6).

v. Repeat (ii) - (iv) N times obtaining the estimates $\tilde{k}_1, \ldots, \tilde{k}_N$.

vi. Calculate MB and MSE using

$$MB = \frac{1}{N} \left| \sum_{i=1}^{N} (\tilde{k}_i - k) \right|$$

$$MSE = \frac{1}{N} \sum_{i=1}^{N} (\tilde{k}_i - k)^2$$

The computer program for this simulation was written in GWBASIC and is reproduced as Program (1) in the Appendix. Some of the simulation values are given in Tables (2.2.1)-(2.2.7).

From these tables we can conclude that

Given k, a, R, N, the MB and MSE of $\tilde{k}$ decreases as SS increases.

2.2.3 a and k are Parameters

As before we have

$$L(a,k) = \sum_{i=1}^{r} \left(\frac{a}{k+1} t_i^{k+1} - \xi_i \right)^2$$

where both a and k are parameters.

Therefore to obtain the values of a and k that minimize L(a,k) we have

$$\frac{\partial L(a,k)}{\partial k} = 2 \sum_{i=1}^{r} \left(\frac{a}{k+1} t_i^{k+1} - \xi_i \right) \cdot \frac{a}{k+1} t_i^{k+1} \qquad (2.2.8)$$

$$\frac{\partial L(a,k)}{\partial k} = \frac{2a}{(k+1)^2} \sum_{i=1}^{r} \left[\frac{a}{k+1} t_i^{k+1} - \xi_i\right] \left[t_i^{k+1} \ln t_i (k+1) - t_i^{k+1}\right] \qquad (2.2.9)$$

Furthermore,

$$\frac{\partial^2 L(a,k)}{\partial a^2} = \frac{2F}{(k+1)^2}$$

$$\frac{\partial^2 L(a,k)}{\partial a \partial k} = \frac{2a}{(k+1)^4} \left[2(k+1)^2 E - 2(k+1)F\right] - \frac{2}{(k+1)^2} \left[B(k+1) - C\right]$$

$$\frac{\partial^2 L(a,k)}{\partial k^2} = \frac{-2a}{(k+1)^3} \left[A(k+1)^2 - 2 (B(k+1) - C) \right]$$

$$+ \frac{2a^2}{(k+1)^4} \left[(2D(k+1)^2 - F) - 4 (E(k+1) - F) \right]$$

where A, B, C, D, E and F are as defined earlier. Then the Hessinian H will be

$$H = \begin{bmatrix} \dfrac{\partial^2 L(a,k)}{\partial a^2} & \dfrac{\partial^2 L(a,k)}{\partial a \partial k} \\ \dfrac{\partial^2 L(a,k)}{\partial a \partial k} & \dfrac{\partial^2 L(a,k)}{\partial k^2} \end{bmatrix}$$

Therefore,

$$H = \frac{2F}{(k+1)^2} \left\{ \frac{2a^2}{(k+1)^4} \left[(2D(k+1)^2 - F) - 4\,[E(k+1) - F] \right] - \frac{2a}{(k+1)^3} \left[A(k+1)^2 - 2(B(k+1) - C) \right] \right\} - \left\{ \frac{2a}{(k+1)^4} \cdot \left[2(k+1)^2 E - 2(k+1)F \right] - \frac{2}{(k+1)^2} \left[B(k+1) - C \right] \right\}^2$$

To find the least square type estimates of a and k equate to zero equation (2.2.8) and (2.2.9). Let $\tilde{a}$ and $\tilde{k}$ be the solution (if they exists), of these equations for a and k respectively. If

$$H \Big|_{a=\tilde{a},\,k=\tilde{k}} > 0 \quad \text{and} \quad \frac{\partial^2 L(a,k)}{\partial a^2} \Big|_{a=\tilde{a},\,k=\tilde{k}} > 0$$

then $\tilde{a}$ and $\tilde{k}$ will be the least squares type estimates of a and k respectively. It is obvious that $\dfrac{\partial^2 L(a,k)}{\partial a^2} \Big|_{a=\tilde{a},\,k=\tilde{k}} > 0$.

But we have to show numerically that

$$H \Big|_{a=\tilde{a},k=\tilde{k}} > 0 \qquad (2.2.10)$$

We don't know whether the least squares type estimates of a and k are unbiased or not and also their variances are not known. Therefore, we want to study these quantities by simulation. We mentioned earlier how to obtain a random sample from the density function (2.2.2).

The simulation procedure is as follows:

i. Fix the values of N, a, k, SS and R.

ii. Obtain a value of U and calculate t using the relation (2.2.7).

iii. Repeat the process (ii) SS times. This gives us the sample $t_1, \ldots, t_{SS}$ then take the smallest R observations.

iv. Obtain the least squares type estimates $\tilde{a}$ and $\tilde{k}$ of a and k respectively using the equations (2.2.8) and (2.2.9) and then check the condition in inequality (2.2.10).

v. Repeat (ii)-(iv) N times obtaining the LSE's $\tilde{a}_1, \ldots, \tilde{a}_N$ and $\tilde{k}_1, \ldots, \tilde{k}_N$.

vi. Calculate mean bias of a (MB(a)), mean bias of (MB(k)), mean squares error of a (MSE(a)) and mean squares error of (MSE(k)), where

$$MB(a) = \frac{1}{N} \left| \sum_{i=1}^{N} (\tilde{a}_i - a) \right|$$

$$MB(k) = \frac{1}{N} \left| \sum_{i=1}^{N} (\tilde{k}_i - k) \right|$$

$$MSE(a) = \frac{1}{N} \sum_{i=1}^{N} (\tilde{a}_i - a)^2$$

$$MSE(k) = \frac{1}{N} \sum_{i=1}^{N} (\tilde{k}_i - k)^2$$

The computer program for this simulation is written in GWBASIC and is reproduced as Program (2) in the Appendix. The simulation values are given in Tables (2.2.8)-(2.2.14).

From the tables we can conclude that

Given a, k, R, N, the MB(k), MSE(k), MB(a) and MSE(a) decreases as SS increases.

Also one might conclude that the least squares type estimates of k is biased whereas the least square type estimate of a is unbiased.

Table 2.2.1

N= 100 a= 5 R=INT(SS/2)

k	SS	MB	MSE
1	10	2.67595	8.83092
1	20	1.75399	3.53421
1	30	1.1566	1.48123
1	40	.895638	.872863
1	50	.702549	.56095

Table 2.2.2

N= 200 a= 5 R=INT(SS/2)

k	SS	MB	MSE
1	10	3.55848	17.055
1	20	2.41132	6.85837
1	30	1.90114	3.86218
1	40	1.61481	2.80591
1	50	1.35969	1.96604

Table 2.2.3

N= 300 a= 5 R=INT(SS/2)

k	SS	MB	MSE
1	10	3.95504	19.9851
1	20	2.94293	9.68666
1	30	2.41348	6.22026
1	40	2.07984	4.56991
1	50	1.78578	3.34

Table 2.2.4

N= 100	A= 1	R=INT(SS/2)	
k	**SS**	**MB**	**MSE**
2	10	4.49185	45.4926
2	20	3.2448	18.5838
2	30	2.53994	8.39007
2	40	2.25679	6.26148
2	50	1.99306	4.53396

Table 2.2.5

N= 100	a= 1	R=INT(SS/2)	
k	**SS**	**MB**	**MSE**
1	10	5.60798	63.4713
1	20	4.68339	29.6303
1	30	3.2474	12.5633
1	40	2.59222	8.24552
1	50	2.00767	4.71558

Table 2.2.6

N= 100	a= 2	R=INT(SS/2)	
k	**SS**	**MB**	**MSE**
1	10	4.93426	58.3480
1	20	3.92401	27.2489
1	30	2.58473	9.45019
1	40	1.73657	3.96195
1	50	1.21081	1.75488

Table 2.2.7

N= 100	a= 2	R=INT(3*SS/4)	
k	SS	MB	MSE
1	10	7.14861	94.4457
1	20	4.89413	32.4811
1	30	4.10714	19.2232
1	40	3.24456	11.5271
1	50	2.27382	5.85237

Table 2.2.8

N= 100	k= 1	a= .0005	R=INT(SS/4)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.249	0.62699	0.00385	0.0000489
20	0.154	0.38940	0.00199	0.0000278
30	0.158	0.19312	0.00129	0.0000092
40	0.040	0.24141	0.00077	0.0000043
50	0.113	0.08262	0.00051	0.0000013

Table 2.2.9

N= 200	k= 1	a= .0005	R=INT(SS/4)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.255	0.46841	0.00352	0.0000540
20	0.151	0.26958	0.00143	0.0000116
30	0.098	0.18778	0.00085	0.0000047
40	0.081	0.16534	0.00078	0.0000037
50	0.079	0.09622	0.00046	0.0000013

Table 2.2.10

N= 100	k= 1	a= .0001	R=INT(SS/4)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.312	0.36324	0.00193	0.0000193
20	0.235	0.22832	0.00073	0.0000030
30	0.138	0.13159	0.00038	0.0000011
40	0.116	0.09115	0.00031	0.0000012
50	0.103	0.07173	0.00016	0.0000002

Table 2.2.11

N= 100	k= 1	a= .01	R=INT(SS/3)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.840	0.86562	0.02726	0.0010218
20	0.562	0.44307	0.01562	0.0004455
30	0.541	0.37873	0.01540	0.0003720
40	0.447	0.28156	0.01222	0.0002653
50	0.371	0.20064	0.00910	0.0001550

Table 2.2.12

N= 100	k= 1	a= .01	R=INT(2*SS/3)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.586	0.43897	0.02448	0.0010617
20	0.429	0.26036	0.01642	0.0004954
30	0.337	0.14830	0.01024	0.0001722
40	0.295	0.12697	0.00895	0.0001540
50	0.276	0.11041	0.00824	0.0001257

Table 2.2.13

N= 100	k= 1	a= .1	R=INT(2*SS/3)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.598	0.45533	0.05885	0.0061684
20	0.391	0.21044	0.04342	0.0036300
30	0.364	0.17401	0.04158	0.0029534
40	0.299	0.12275	0.03311	0.0020418
50	0.269	0.10531	0.02705	0.0015495

Table 2.2.14

N= 100	k= 2	a= .1	R=INT(2*SS/3)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	0.906	1.06281	0.07169	0.0093096
20	0.665	0.61670	0.05678	0.0057191
30	0.494	0.35339	0.04534	0.0043734
40	0.458	0.30627	0.04239	0.0033541
50	0.402	0.23719	0.03585	0.0024263

CHAPTER THREE

Maximum Likelihood Estimates

In Chapter Two we obtained the least squares type estimates of the coefficients in the linear hazard function and the least squares type estimates of the parameters for the power hazard function. In this chapter we obtain the maximum likelihood estimates of the same parameters. In Section (3.1) we describe briefly the method of the maximum likelihood estimation of the parameters. The results obtained by Bain (1974) for the linear hazard function are briefly described in Section (3.2) and we describe in the same section the results of the simulated study for these parameters. In Section (3.3) we give Bain's (1974) expression for the variance-covariance matrix for the maximum likelihood estimates of the parameters of the linear hazard function. In Section (3.4) we give the maximum likelihood estimates of the parameters for the power hazard function; also included are some simulation results. The maximum likelihood estimates are given using an r censored sample; we take r = n if we are using a complete sample.

With respect to the polynomial hazard function it was not possible to give explicit expressions for the maximum likelihood estimates of the parameters. These extimates can be obtained using some numerical approximation technique.

3.1 Maximum Likelihood Estimates

Suppose that the random variable T has probability density function $f(t|\theta)$. For a random sample $t_1, \ldots, t_n$ the joint density function is given by

$$f\left(t_1, \ldots, t_n|\theta\right) = \prod_{i=1}^{n} f\left(t_i|\theta\right)$$

For fixed $t_1, \ldots, t_n$ the joint density function as a function of θ is called the likelihood function of the sample and is usually denoted by $L(\theta|t_1,\ldots,t_n)$.

The maximum likelihood estimate of θ is the value of θ (as a function of the sample values $t_1,\ldots,t_n$) for which the likelihood function $L(\theta|t_1,\ldots t_n)$ attains its maximum value. Usually the maximum likelihood estimate of θ exists and is unique.

3.2 Linear Hazard Function

Now let us consider the linear hazard function $h(t) = a + bt$. Using equation (1.1.5) we get

$$F(t) = 1 - \exp\left[-\left(at + \frac{bt^2}{2}\right)\right], \qquad t > 0$$

Therefore, the density function $f(t)$ is given by

$$f(t) = (a + bt)\exp\left[-\left(at + \frac{bt^2}{2}\right)\right], \qquad t > 0 \qquad (3.2.1)$$

As mentioned earlier, this case was studied by Bain (1974) in detail. Hence we describe his results briefly.

Suppose $t_1, \ldots, t_r$ denote the smallest r observations from a random sample of size n having the density function (3.2.1). Therefore the likelihood function of the sample is given by

$$L(a,b|n,t_1,\ldots,t_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} (a+bt_i) \cdot$$

$$\exp\left[-\left[\sum_{i=1}^{r}\left(at_i + \frac{bt_i}{2}\right) + (n-r)\left(at_r + \frac{bt_r^2}{2}\right)\right]\right]$$

Since the likelihood function is differentiable with respect to a and b, the maximum likelihood estimates of the parameters a and b can be determined by equating to zero the partial derivatives of the log-likelihood function with respect to a and b:

$$\frac{\partial \ell n L}{\partial a} = \sum_{i=1}^{r} \frac{1}{a+bt_i} - \left[\sum_{i=1}^{r} t_i + (n-r)\ t_r\right] = 0$$

$$\frac{\partial \ell n L}{\partial b} = \sum_{i=1}^{r} \frac{t_i}{a+bt_i} - \frac{1}{2}\left[\sum_{i=1}^{r} t_i^2 + (n-r)\ t_r^2\right] = 0$$

Let $u_k = \sum_{i=1}^{r} t_i^k + (n-r)\ t_r^k , \quad k = 1, 2$

This leads to the equations

$$\sum_{i=1}^{r} \frac{1}{\hat{a}+\hat{b}t_i} = u_1$$

and

$$\sum_{i=1}^{r} \frac{t_i}{\hat{a}+\hat{b}t_i} = u_2 / 2$$

Also, we notice that

$$\hat{a}u_1 + \hat{b}u_2 / 2 = r$$

Suppose we restrict a and b to positive values. Since

$$a\,u_1 + \frac{bu_2}{2} = r$$

this means that

$$0 \le a \le r / u_1 \quad \text{and} \quad 0 \le b \le 2r / u_2 .$$

Now considering a as a differentiable function of b we can write

$$a(b)\,u_1 + \frac{bu_2}{2} = r$$

Imposing this relationship on the likelihood function yields

$$L(b/n,t_1,\ldots,t_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} (a(b)+ bt_i)\, e^{-r} .$$

This gives

$$g(b) = \frac{\partial \ell n L}{\partial b} = \sum_{i=1}^{r} \left\{ \left(\frac{\partial a}{\partial b} + t_i \right) / (a(b) + bt_i) \right\}$$

But $a(b) = \dfrac{2r - bu_2}{2u_1}$

and $\dfrac{\partial a}{\partial b} = \dfrac{-u_2}{2\,u_1}$

Therefore,

$$g(b) = \sum_{i=1}^{r} \frac{t_i - \dfrac{u_2}{2u_1}}{\dfrac{2r-bu_2}{2u_1} + bt_i} = \sum_{i=1}^{r} \frac{1}{b+r/(u_1 t_i-(u_2/2))}$$

Also, assuming g(b) to be a differentiable function of b in the region of interest we obtain

$$g'(b) = -\sum_{i=1}^{r} \left\{ \left(t_i + \frac{\partial a}{\partial b}\right)^2 \Big/ (a(b) + bt_i)^2 \right\} < 0$$

This implies that g(b) is a monotonically decreasing function of b and therefore can have atmost one zero in the region. If such a zero exists then the likelihood function attains the maximum value at that zero point.

But $0 \le b \le 2r/u_2$ and g(b) is a monotonically decreasing and continuous function of b. If $g(2r/u_2) \ge 0$ then we take $\hat{b} = 2r/u_2$ and $\hat{a} = 0$. If $g(0) \le 0$ then we take $\hat{b} = 0$ and $\hat{a} = r/u_1$. If $g(2r/u) < 0$ and $g(0) > 0$ then $\hat{b}$ is the unique solution of $g(b) = 0$ which can be determined by using, say, Newton-Raphson method.

We don't know whether the maximum likelihood estimates of the parameters a and b are unbiased or not. Also we don't know their variances. Therefore, we want to study these quantities by using simulation.

The first step to simulation is to obtain a random sample from the density function (3.2.1). The procedure for it is as follows:

i. Obtain the distribution function

$$F(t) = 1 - \exp[-(at + bt^2/2)] \quad , \quad t > 0.$$

ii. If U is a random number in [0,1) then equate

$$U = 1 - \exp[-(at + bt^2/2)] \quad , \quad t > 0.$$

This gives

$$t = \frac{-a + \sqrt{a^2 - 2b\ln(1-U)}}{b} \tag{3.2.2}$$

The other value of t is not used because it is negative.

The following symbols will be used.

N : number of samples

A, B : Parameters a and b respectively

SS : Sample size

U : Random number in [0,1)

MBA : Mean bias of A

MSEA : Mean squared error of A

MBB : Mean bias of B

MSEB : Mean squared error of B

R : Number of failed units at which the experiment is terminated.

The simulation procedure is as follows:

i. Fix the values of A, B, N and SS.

ii. Obtain a value of U and calculate t using the relation (3.2.2)

iii. Repeat the process (ii) SS times. This gives us the sample $t_1, \ldots, t_{SS}$. Then take the smallest r observations.

iv. Obtain the maximum likelihood estimates of A and B as described in this section.

v. Repeat (ii)-(iv) N times obtaining the maximum likelihood estimates $\hat{A}_1, \ldots, \hat{A}_N$ and $\hat{B}_1, \ldots, \hat{B}_N$.

vi. Calculate MBA, MBB, MSEA and MSEB where

$$MBA = \frac{1}{N} \left| \sum_{i=1}^{N} \left(\hat{A}_i - A \right) \right|$$

$$MBB = \frac{1}{N} \left| \sum_{i=1}^{N} \left(\hat{B}_i - B \right) \right|$$

$$MSEA = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{A}_i - A \right)^2$$

$$MSEB = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{B}_i - B \right)^2$$

The computer program for this simulation is written in GWBASIC and is reproduced as Program (3) in the Appendix. The simulated values are given in Tables (3.2.1) and (3.2.2).

From the tables we can conclude that

i. Given N, A and B, MBA, MBB, MSEA and MSEB decreases as SS increases.

ii. Given N, R and A, MBB and MSEB increases as B increases.

iii. Given N, R and B, MBA and MSEA increases as A increases.

3.3 Asymptotic Variance-Covariance Matrix

These results are also given by Bain (1974). The asymptotic variance-covariance matrix for the maximum likelihood estimates of a and b is given by $V = (1/n)\ A^{-1}$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with

$$a_{11} = -\lim_{n\to\infty} E\left\{\frac{1}{n}\frac{\partial^2 \ln f}{\partial a^2}\right\}$$

$$a_{12} = -\lim_{n\to\infty} E\left\{\frac{1}{n}\frac{\partial^2 \ln f}{\partial a\partial b}\right\}$$

$$a_{22} = -\lim_{n\to\infty} E\left\{\frac{1}{n}\frac{\partial^2 \ln f}{\partial b^2}\right\}$$

and $$\lim_{n\to\infty}\frac{r}{n} = p$$

But

$$f(t_1,\ldots,t_r|a,b,n) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}(a+bt_i)\ e^{-r}$$

Therefore,

$$\ln f = \ln\frac{n!}{(n-r)!} + \sum_{i=1}^{r}\ln(a+bt_i)$$

and

$$\frac{\partial \ln f}{\partial a} = \sum_{i=1}^{r} \frac{1}{a+bt_i}$$

and

$$\frac{\partial^2 \ln f}{\partial a^2} = -\sum_{i=1}^{r} \frac{1}{(a+bt_i)^2} = -\frac{1}{a^2}\sum_{i=1}^{r} \frac{a^2}{(a+bt_i)^2}$$

Also, we have

$$\frac{\partial^2 \ln f}{\partial a \partial b} = -\sum_{i=1}^{r} \frac{t_i}{(a+bt_i)^2}$$

$$= \frac{-1}{ab}\left[\sum_{i=1}^{r} \frac{a}{a+bt_i} - \sum_{i=1}^{r} \frac{a^2}{(a+bt_i)^2}\right]$$

and therefore

$$\frac{\partial^2 \ln f}{\partial b^2} = -\sum_{i=1}^{r} \frac{t_i^2}{(a+bt_i)^2}$$

$$= \frac{-1}{b^2}\left[r + \sum_{i=1}^{r} \frac{a^2}{(a+bt_i)^2} - 2\sum_{i=1}^{r} \frac{a}{a+bt_i}\right]$$

Bain (1974) has shown that

$$a_{11} = A(\beta,p) \,/\, a^2$$

$$a_{12} = [\, B(\beta,p) - A(\beta,p) \,] \,/\, ab$$

and

$$a_{22} = [\, p + A(\beta,p) - 2B(\beta,p) \,] \,/\, b^2$$

where

$$\beta = b \,/\, a^2$$

$$A(\beta,p) = \frac{1}{2\beta}\, e^{1/2\beta}\left[E(1/2\beta) - E\left[(1/2\beta) - \ln(1-p) \right] \right]$$

$$E(x) = \int_x^\infty \frac{e^{-y}}{y}\, dy$$

$$B(\beta,p) = \left(\frac{2\pi}{\beta}\right)^{1/2} e^{1/\beta}\left[N\left[\left((1/\beta)-2\ell n\ (1-p)\right)^{1/2}\right] - N\left(\beta^{-1/2}\right)\right]$$

and N(x) denotes the distribution function of the standard normal distribution.

3.4 Power Hazard Function

In this section we give the maximum likelihood estimates of the parameters of the power hazard function $h(t) = a\, t^k$ in three casses (i) a is the parameter and k is known, (ii) k is the parameter and a is known and (iii) a and k are parameters. This section has also some simulation results.

3.4.1 a is the Parameter and k is Known

Suppose $t_1, \ldots, t_r$ denote the smallest r observations of a sample of size n from the density function of the power hazard function $h(t) = at^k$. Then the joint density function is given by

$$f = f(t_1,\ldots,t_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} (at_i^k) \exp\left[-\frac{a}{k+1}\left(\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1}\right)\right]$$

Then to determine the maximum likelihood estimate of a consider

$$\ln f = \ln \frac{n!}{(n-r)!} + r \ln a + k \sum_{i=1}^{r} \ln t_i - \frac{a}{k+1} \left[\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} \right]$$

and therefore

$$\frac{\partial \ln f}{\partial a} = \frac{r}{a} - \frac{1}{k+1} \left[\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} \right].$$

$\frac{\partial \ln f}{\partial a} = 0$ gives the unique solution

$$\hat{a} = r\,(k+1) \Big/ \left[\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} \right]$$

Is $\hat{a}$ an unbiased estimator of a ?

It is easily seen that

$$t^{k+1} \sim \text{EXP}\left(\frac{k+1}{a}\right)$$

But

$$\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} = \sum_{i=1}^{r} (n-i+1) \left(t_{(i)}^{k+1} - t_{(i-1)}^{k+1}\right)$$

with $t_{(0)} = 0$

Let $Z_i = (n-i+1) \left(t_{(i)}^{k+1} - t_{(i-1)}^{k+1}\right) \qquad i = 1, \ldots, r$

Then $Z_1, \ldots, Z_r$ are iid exp $\left(\frac{k+1}{a}\right)$ (Sinha, (1986)).

Hence

$$\sum_{i=1}^{r} Z_i = \sum_{i=1}^{r} (n-i+1) \left(t_{(i)}^{k+1} - t_{(i-1)}^{k+1}\right) \sim \Gamma\left(r, \frac{k+1}{a}\right).$$

Let $\qquad Y = \sum_{i=1}^{r} Z_i$

But $E\,\hat{a} = r(k+1)\,E\,\frac{1}{Y}$.

So, we calculate

$$E\,\frac{1}{Y} = \int_0^\infty \frac{1}{\Gamma(r)}\left(\frac{a}{k+1}\right)^r e^{-ay/(k+1)}\, y^{r-2}\, dy$$

$$= \frac{a}{k+1} \cdot \frac{1}{r-1} \quad \text{if } r > 1.$$

Finally, we have

$$E\,\hat{a} = \frac{ra}{r-1} \quad \text{if } r > 1.$$

We notice that $\hat{a}$ is a biased estimator of a.

Consider

$$\tilde{a} = \frac{r-1}{r}\,\hat{a} = \frac{(r-1)\,(k+1)}{\sum_{i=1}^{r} t_i^{k+1} + (n-r)\,t_r^{k+1}} \quad \text{if } r > 1.$$

It is clear seen that $\tilde{a}$ is an unbiased estimator of a.

Also,

$$\mathrm{Var}(\tilde{a}) = (r-1)^2\,(k+1)\,\mathrm{Var}(1/y)$$

But

$$\mathrm{Var}\left(\frac{1}{y}\right) = \left(\frac{a}{k+1}\right)^2 \frac{1}{(r-1)^2\,(r-z)} \quad \text{if } r > 2.$$

Therefore,

$$\mathrm{Var}(\tilde{a}) = \frac{a^2}{r-2} \quad \text{if } r > 2.$$

Since Y is a complete sufficient statistic for a, therefore, $\tilde{a}$ is also the unique uniformly minimum variance unbiased estomator of a.

3.4.2 k is the Parameter and a is Known

Suppose $t_1, \ldots, t_r$ denote the smallest r observations from a sample of size n from the density function having the power hazard function $h(t) = at^k$. Then the joint density is given by

$$f = \frac{n!}{(n-r)!} \prod_{i=1}^{r} at_i^k \exp\left[-\frac{a}{k+1}\left(\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1}\right)\right]$$

Therefore,

$$\ln f = \ln \frac{n!}{(n-r)!} + \sum_{i=1}^{r} \ln a + k \sum_{i=1}^{r} \ln t_i - \frac{a}{k+1}\left[\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1}\right]$$

Hence

$$\frac{d\ln f}{\partial k} = \sum_{i=1}^{r} \ln t_i - \frac{a}{(k+1)^2}\left[\left(\sum_{i=1}^{r} t_i^{k+1} \ln t_i + (n-r) t_r^{k+1} \ln t_r\right) . (k+1) - \left(\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1}\right)\right] .$$

In order to obtain the maximum likelihood estimator of k we equate $\frac{d\ln f}{\partial k}$ to zero

$$0 = \frac{d\ln f}{\partial k} = \sum_{i=1}^{r} \ln t_i - \frac{a}{(k+1)^2}\left[\left(\sum_{i=1}^{r} t_i^{k+1} \ln t_i + (n-r) t_r^{k+1} \ln t_r\right) . (k+1) - \left(\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1}\right)\right] . \quad (3.4.1)$$

Let

$$g(k) = \frac{d\ln f}{\partial k}$$

We want to know if there exists a solution of k for equation (3.4.1), such that $-1 < k$. It is easily seen that

$$\lim_{k \to -1} g(k) = \infty$$

$$\lim_{k \to \infty} g(k) = \sum_{i=1}^{r} \ln t_i - \lim_{k \to \infty} a \frac{g_1(k)}{g_2(k)}$$

where

$$g_1(k) = \left[\left(\sum_{i=1}^{r} t_i^{k+1} \ln t_i + (n-r) t_r^{k+1} \ln t_r\right) (k+1) - \left(\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1}\right)\right] .$$

and $g_2(k) = (k+1)^2$.

Therefore, $\lim_{k \to \infty} \frac{g_1(k)}{g_2(k)}$ is of the form $\pm\frac{\infty}{\infty}$ unless $\max t_i < 1$, $i = 1, \ldots, r$ $\lim_{k \to \infty} g(k) = \Sigma \ln t_i < 0$. So by L'Hospital rule

$$\lim_{k \to \infty} \frac{g_1(k)}{g_2(k)} = \lim_{k \to \infty} \frac{g_1'(k)}{g_2'(k)}$$

But

$$g_1'(k) = \left[\sum t_i^{k+1} \left(\ln t_i\right)^2 + (n-r)\ t_i^{k+1} \left(\ln t_r\right)^2 \right] (k+1)$$

and

$$g_2'(k) = 2\ (k+1)\ .$$

Therefore,

$$\frac{g_1'(k)}{g_2'(k)} = \frac{1}{2} \left[\sum t_i^{k+1} \left(\ln t_i\right)^2 + (n-r)\ t_r^{k+1} \left(\ln t_r\right)^2 \right] .$$

Hence

$$\lim_{k \to \infty} g(k) = \sum \ell n t_i - \lim_{k \to \infty} \frac{a}{2} \left[\sum_{i=1}^{r} t_i^{k+i} \left(\ell n t_i \right)^2 + (n-r) t_r^{k+1} \left(\ell n t_r \right)^2 \right]$$

which is negative.

Since g(k) is monotonically decreasing and is a continuous function then there exists a solution for equation (3.4.1) and the solution is unique. Let $\hat{k}$ be the solution of the equation (3.4.1).

Furthermore,

$$\frac{d^2 \ell n f}{dk^2} = \frac{-a}{(k+1)^4} \left[\left((A+B)(k+1) + C + D - C - D \right) (k+1)^2 - 2(k+1) \left[(C+D)(k+1) - (E+F) \right] \right]$$

where

$$A = \sum_{i=1}^{r} t_i^{k+1} \left(\ell n t_i \right)^2$$

$$B = (n-r)\, t_r^{k+1} \left(\ell n t_r \right)^2$$

$$C = \sum_{i=1}^{r} t_i^{k+1}\, \ell n t_i$$

$$D = (n-r)\, t_r^{k+1}\, \ell n t_r$$

$$E = \sum_{i=1}^{r} t_i^{k+1}$$

$$F = (n-r)\, t_r^{k+1}, \quad G = \sum_{i=1}^{r} \ell n\, t_i$$

Therefore,

$$\frac{d^2 \ell n f}{dk^2} = \frac{-a}{(k+1)^3} \left[\left\{ (A+B)\ (k+1)^2 - 2 \left[(C+D)\ (k+1) - (E+F) \right] \right] \right.$$

But from equation (3.4.1).

$$G \frac{(k+1)^2}{a} = (C+D)\ (k+1) - (E+F)$$

Therefore

$$\frac{d^2 \ell n f}{dk^2} = \frac{-a}{(k+1)^3} \left[(A+B)\ (k+1)^2 - 2 \frac{(k+1)^2}{a} G \right]$$

$$= \frac{-a}{(k+1)} [A+B] + \frac{2G}{k+1} .$$

Furthermore

$$\left. \frac{d^2 \ell n f}{dk^2} \right|_{k=\hat{k}} = \frac{-a}{(k+1)} [\ \hat{A}+\hat{B}\ ] + \frac{2G}{k+1}$$

where

$$\hat{A} = \sum_{i=1}^{r} t_i^{\hat{k}+1} \left(\ell n t_i\right)^2$$

and

$$\hat{B} = (n-r)\ t_r^{\hat{k}+1} \left(\ell n t_r\right)^2$$

Therefore,

$$\left. \frac{d^2 \ell n f}{dk^2} \right|_{k=\hat{k}} < 0$$

iff

$$a > \frac{2\ G}{(A+B)} . \qquad (3.4.2)$$

Therefore, when the condition (3.4.2) is satisfied the maximum likelihood estimate $\hat{k}$ of k is the solution of equation (3.4.1).

We don't know whether the maximum likelihood estimate of the parameter k is unbiased or not and also it's variance is unknown. Therefore, we want to study these quantities by simulation.

The way to simulate from the density function of the power hazard function was discussed in Section (2.2.2), and we use the symbols given in that section.

The simulation procedure is as follows:

i. Fix the values a, k, N, SS and R.

ii. Obtain a value of U and calculate t using the relation (2.4.8).

iii. Repeat the process (ii) SS times. This give us the sample $t_1, \ldots, t_{SS}$ and take the smallest R observations $t_1, \ldots, t_R$.

iv. Obtain the maximum likelihood estimate $\hat{k}$ of k by using the equation (3.4.1) and then check the condition in inequality (3.4.2). If the condition is not satisfied then repeat the sample and repeat the procedure.

v. Repeat (ii)-(iv) N times obtaining the maximum likelihood estimates $\hat{k}_1, \ldots, \hat{k}_N$.

vi. Calculate MB and MSE using

$$MB = \frac{1}{N} \left| \sum_{i=1}^{N} \left(\hat{k}_i - k\right) \right|$$

$$MSE = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{k}_i - k\right)^2$$

The computer program for this simulation is written in GWBASIC and is reproduced as Program (4) in the Appendix. The simulated values are given in Tables (3.4.1)-(3.4.7).

From these tables we can conclude that Given a, k, R, N, the MB and MSE of $\hat{k}$ decreases as SS increases.

3.4.3 a and k are Parameters

Suppose $t_1, \ldots, t_r$ denotes the smallest r observations from a sample of size n from the density function of the power hazard function $h(t) = a\, t^k$. Then the joint density is given by

$$f(t_1,\ldots,t_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} a t_i^k \exp\left[- \frac{a}{k+1} \left(\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1} \right) \right].$$

Therefore,

$$\ell n f = \ell n \frac{n!}{(n-r)!} + \sum_{i=1}^{r} \ell n a + k \sum_{i=1}^{r} \ell n t_i - \frac{a}{k+1} \left[\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1} \right]$$

and

$$\frac{\partial \ln f}{\partial k} = \sum_{i=1}^{r} \ln t_i - \frac{a}{(k+1)^2} \left[\left(\sum_{i=1}^{r} t_i^{k+1} \ln t_i + (n-r)\, t_r^{k+1} \ln t_r \right) \right.$$

$$\left. \cdot\, (k+1) - \left(\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} \right) \right]$$

$$\frac{\partial \ln f}{\partial a} = \frac{r}{a} - \frac{1}{k+1} \left[\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1} \right]$$

To find the maximum likelihood estimates of a and k let

$$\frac{\partial \ln f}{\partial k} = 0 \qquad (3.4.3)$$

$$\frac{\partial \ln f}{\partial a} = 0 \qquad (3.4.4)$$

From (3.4.4) we obtain $a = \left[\dfrac{r\,(k+1)}{\sum_{i=1}^{r} t_i^{k+1} + (n-r)\, t_r^{k+1}} \right]$

Substitute this value of a in equation (3.4.3) to get the equation

$$\sum_{i=1}^{r} \ln t_i - \frac{r \left[\sum_{i=1}^{r} t_i^{k+1} \ln t_i + (n-r) t_r^{k+1} \ln t_r \right]}{\sum_{i=1}^{r} t_i^{k+1} + (n-r) t_r^{k+1}} + \frac{r}{k+1} = 0$$

By solving this equation numerically we get the value $\hat{k}$ (if it exists) which is the maximum likeihood estimate of k. Assuming that this value is unique then we find the maximum likelihood estimate $\hat{a}$ of a by using

$$\hat{a} = \frac{r\,(\hat{k}+1)}{\sum_{i=1}^{r} t_i^{\hat{k}+1} + (n-r)\, t_i^{\hat{k}+1}} .$$

Furthermore,

$$\frac{\partial^2 \ell nf}{\partial a^2} = -\frac{r}{a^2}$$

$$\frac{\partial^2 \ell nf}{\partial a dk} = -\frac{1}{(k+1)^2}\left[(C+D)(k+1) - (E+F)\right]$$

and

$$\frac{\partial^2 \ell nt}{\partial k^2} = -\frac{a}{(k+1)^4}\left[(A+B)(k+1)^3 - 2(k+1)\left[(C+D)(k+1)-(E+F)\right]\right]$$

where

$$A = \sum_{i=1}^{r} t_i^{k+1} (\ell nt_i)^2$$

$$B = (n-r)\ t_r^{k+1}\ \ell nt_r$$

$$C = \sum_{i=1}^{r} t_i^{k+1}\ \ell nt_i$$

$$D = (n-r)\ t_r^{k+1}\ \ell nt_r$$

$$E = \sum_{i=1}^{r} t_i^{k+1}$$

$$F = (n-r)\ t_r^{k+1}$$

The maximum likelihood estimates of a and k exists if

$$H \Big|_{a=\hat{a},k=\hat{k}} > 0 \quad \text{and} \quad \frac{\partial^2 \ell nf}{\partial a^2}\Big|_{a=\hat{a}} < 0$$

Its obvious that $\frac{\partial^2 \ell nf}{\partial a^2}\Big|_{a=\hat{a}} < 0$.

Now

$$H\Big|_{a=\hat{a},k=\hat{k}} = \frac{r}{\hat{a}(\hat{k}+1)^4}\left[(\hat{A}+\hat{B})(\hat{k}+1)^3 - 2(\hat{k}+1)\left[(\hat{C}+\hat{D})(\hat{k}+1)-(\hat{E}+\hat{F})\right]\right]$$

$$- \frac{1}{(k+1)^4}\left[(\hat{C}+\hat{D})(\hat{k}+1)-(\hat{E}+\hat{F})\right]^2 .$$

Therefore,

$$H\Big|_{a=\hat{a},k=\hat{k}} > 0$$

iff

$$\hat{a} < \frac{r\left[(\hat{A}+\hat{B})(\hat{k}+1)^3-2(\hat{k}+1)\left[(\hat{C}+\hat{D})(\hat{k}+1)-(\hat{E}+\hat{F})\right]\right]}{\left[(\hat{C}+\hat{D})(\hat{k}+1)-(\hat{E}+\hat{F})\right]^2} \qquad (3.4.5)$$

where

$$\hat{A} = \sum_{i=1}^{r} t_i^{\hat{k}+1}(\ell n t_i)^2$$

$$\hat{B} = (n-r)\, t_r^{\hat{k}+1}\, \ell n t_r$$

$$\hat{C} = \sum_{i=1}^{r} t_i^{\hat{k}+1}\, \ell n t_i$$

$$\hat{D} = (n-r)\, t_r^{\hat{k}+1}\, \ell n t_r$$

$$\hat{E} = \sum_{i=1}^{r} t_i^{\hat{k}+1}$$

$$\hat{F} = (n-r)\, t_r^{\hat{k}+1}$$

Therefore, when the condition (3.4.5) is satisfied then the maximum likelihood estimates $\hat{a}$ and $\hat{k}$ of a and k respectively are the solutions of the simultaneous equations (3.4.3) and (3.4.4).

Again we don't know whether the maximum likelihood estimates of a and k are unbiased or not and also their variances are not known. Therefore we want to study these quantities by simulation. We mentioned earlier how to obtain a raondom sample from the density function (2.4.2).

The simulation procedure is as follows:

i. Fix the values of N, a, k, SS and R .

ii. Obtain a value of U and calculate t using the relation (2.4.8).

iii. Repeat the process (ii) SS times. This give us the sample $t_1, \ldots, t_{SS}$. Then take the smallest R observations $t_1, \ldots, t_R$.

iv. Obtain the maximum likelihood estimates $\hat{a}$ and $\hat{k}$ of a and k respectively as we have shown in this section.

v. Repeat (ii)-(iv) N times obtaining the maximum likelihood estimates $\hat{a}_1, \ldots, \hat{a}_N$ and $\hat{k}_1, \ldots, \hat{k}_N$.

vi. Calculate Mean bias of a, (MB(a)), Mean bias of k, (MB(k)), Mean squares error of a, MSE(a) and Mean squares error of k, (MSE(k)

$$MB(a) = \frac{1}{N} \left| \sum_{i=1}^{N} (\hat{a}_i - a) \right|$$

$$MB(k) = \frac{1}{N} \left| \sum_{i=1}^{N} (\hat{k}_i - k) \right|$$

$$MSE(a) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{a}_i - a\right)^2$$

$$MSE(k) = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{k}_i - k\right)^2$$

The computer program for this simulation is also written in GWBASIC and is reproduced as Program (5) in the Appendix. The simulation values are given in Tables (3.4.4) and (3.4.5) respectively.

From these tables we conclude:

Given a, k, R, N, the MB(a), MSE(a), MB(k) and MSE(k) decreases as SS increases.

Table 3.2.1

N= 1000 R=INT(.7*SS)

A= 1 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.284143	1.40276	.357405	10.5911
20	.195389	.740411	.181303	1.96358
30	.139065	.457485	.112	.600292

A= 1 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.277668	1.86671	.525858	19.1231
20	.153906	.866566	.270738	3.82122
30	.137318	.724273	.216267	2.62873

A= 1 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.213164	2.37331	.580494	44.9461
20	.134404	1.07091	.366868	8.20506
30	.09196	.734122	.270761	4.70463

A= 2 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.592092	5.73038	1.51779	138.084
20	.387559	2.81568	.731793	24.4477
30	.335418	2.04267	.489438	12.1864

CONTINUE Table 3.2.1

A= 2 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.515855	6.63465	1.82508	226.996
20	.395795	3.42128	.895257	43.1167
30	.243499	2.11779	.547875	17.5829

A= 2 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.559278	6.36548	1.79218	144.269
20	.323089	3.64121	1.02656	65.8557
30	.23816	2.17136	.604718	24.4246

A= 3 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.801528	13.3432	3.92598	599.98
20	.61322	6.5919	1.82074	136.676
30	.451406	4.29849	1.04862	54.9976

A= 3 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.805714	14.7339	3.42026	3155.2
20	.491855	6.4969	1.7493	139.111
30	.395691	4.60722	1.16909	71.7819

A= 3 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.736848	14.261	4.1716	997.862
20	.624649	6.76189	1.84144	173.721
30	.396562	4.68352	1.24205	82.1644

Table 3.2.2

N= 500 R=INT(.7*SS)

A= 1 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.293306	1.22733	.381531	5.39943
20	.197514	.739022	.17998	1.8816
30	.1364	.529207	.123275	.852704

A= 1 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.283026	1.78649	.444528	14.1968
20	.153307	.947735	.300115	4.43944
30	.130435	.625034	.211274	2.30331

A= 1 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.152805	2.26465	.787833	34.0414
20	.17132	1.13444	.373679	7.0913
30	.772532E-01	.641126	.257003	4.15408

A= 2 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.557358	5.70087	1.45879	178.177
20	.373997	2.71509	.753217	24.0708
30	.275447	1.87339	.490115	10.6054

A= 2 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.479398	6.67028	1.82937	302.6
20	.381826	2.99174	.809281	31.2055
30	.232162	1.85385	.53567	17.4731

Contnue Table 3.2.2

A= 2 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.52221	7.04464	1.81123	173.167
20	.346719	3.37105	.889503	41.7682
30	.236924	2.17489	.656727	23.4374

A= 3 B= 0

SS	MBA	MBB	MSEA	MSEB
10	.949436	14.5418	3.32658	795.435
20	.684372	6.64975	1.78897	129.25
30	.421953	4.5038	1.07257	54.923

A= 3 B= 1

SS	MBA	MBB	MSEA	MSEB
10	.846498	13.5011	3.42466	596.055
20	.505468	6.26599	1.64451	148.109
30	.41539	4.72441	1.12386	73.2327

A= 3 B= 2

SS	MBA	MBB	MSEA	MSEB
10	.855613	15.5104	3.7544	1184.97
20	.503475	6.41837	1.96916	141.765
30	.356542	4.33074	1.30003	81.1112

Table 3.4.1

N=	100	a= 5	R=INT(SS/3)
k	**SS**	**MB**	**MSE**
1	10	3.13899	9.85321
1	20	1.29791	1.68457
1	30	.789272	.622951
1	40	.677823	.459446
1	50	.170397	.290352E-01

Table 3.4.2

N=	200	a= 5	R=INT(SS/3)
k	**SS**	**MB**	**MSE**
1	10	2.34509	5.4995
1	20	1.90841	3.98176
1	30	1.0824	1.17159
1	40	1.31188	1.81028
1	50	1.04266	1.08714

Table 3.4.3

N=	300	a= 5	R=INT(SS/3)
k	**SS**	**MB**	**MSE**
1	10	3.80127	14.4497
1	20	2.33836	5.79122
1	30	2.06798	5.22534
1	40	1.33893	1.7847
1	50	1.18173	1.39648

Table 3.4.1

N= 100	a= 5	R=INT(SS/3)	
k	SS	MB	MSE
1	10	3.13899	9.85321
1	20	1.29791	1.68457
1	30	.789272	.622951
1	40	.677823	.459446
1	50	.170397	.290352E-01

Table 3.4.2

N= 200	a= 5	R=INT(SS/3)	
k	SS	MB	MSE
1	10	2.34509	5.4995
1	20	1.90841	3.98176
1	30	1.0824	1.17159
1	40	1.31158	1.81025
1	50	1.04266	1.08714

Table 3.4.3

N= 300	a= 5	R=INT(SS/3)	
k	SS	MB	MSE
1	10	3.80127	14.4497
1	20	2.33836	5.79122
1	30	2.06798	5.22534
1	40	1.33593	1.7847
1	50	1.18173	1.39648

Table 3.4.4

N= 100	a= 1	R=INT(3*SS/4)	
k	SS	MB	MSE
1	10	.650814	.42386
1	20	.115423	.133225E-01
1	30	.052729	.118035E-01
1	40	.043506	.694351E-02
1	50	.016828	.387413E-02

Table 3.4.5

N= 100	a= 2	R=INT(3*SS/4)	
k	SS	MB	MSE
1	10	.557621	.310941
1	20	.407808	.220641
1	30	.406707	.190713
1	40	.226386	.512505E-01
1	50	.200667	.402672E-01

Table 3.4.6

N= 100	a= 2	R=INT(3*SS/4)	
k	SS	MB	MSE
2	10	1.07416	1.15381
2	20	.339349	.115158
2	30	.215934	.066367
2	40	.185859	.034543
2	50	.132755	.020987

Table 3.4.7

N= 100	a= 1	R=INT(SS/4)	
k	SS	MB	MSE
1	10	.68473	.468854
1	20	.274875	.808813E-01
1	30	.212863	.483065E-01
1	40	.175851	.309236E-01
1	50	.140548	.197536E-01

Table 3.4.8

N= 100 k= 2	a= 1	R=INT(0.75*SS)		
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	.83682	.992768	.462051	.242114
20	.79309	.954809	.454074	.232399
30	.444095	.19722	.136813	.187179
40	.18466	.101421	.365028E-01	.133246
50	.06362	.081492	.141664E-01	.092087

Table 3.4.9

N= 200 k= 2	a= 1	R=INT(0.75*SS)		
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	1.71574	2.94376	.895878	.802697
20	1.06298	1.12992	.926579	.858551
30	1.01668	1.04087	.219858	.483374E-01
40	.464553	.215809	.061816	.38212E-02
50	.282296	.796912E-01	.152626	.232949E-2

Table 3.4.10

N= 100	k= 1	a= 2	R=INT(0.75*SS)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	.801494	.642391	4.7923	22.9661
20	.222464	.494902E-01	.563853	.31793
30	.134164	.179999E-01	.196239	.385096E-01
40	.082365	.006784	.256096	.655851E-01
50	.015513	.344152E-02	.383817	.014716

Table 3.4.11

N= 200	k= 1	a= 2	R=INT(0.75*SS)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	.4245	.180201	.930588	.865996
20	.643536E-01	.414138E-02	.552458	.605209
30	.591908E-01	.270304E-02	.236904	.561237E-01
40	.337296E-01	.113769E-02	.244064	.545674E-01
50	.01267	.916089E-03	.478081	.228561E-01

Table 3.4.12

N= 100	k= 1	a= 1	R=INT(0.75*SS)	
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	.634181	.465369	.134259	.12456
20	.539298	.299404	.109333	.39537E-01
30	.441781	.195171	.467772E-01	.21881E-01
40	.245251	.601482E-01	.834865E-01	.69699E-02
50	.16238	.263672E-01	.252826E-01	.63921E-02

Table 3.3.13

N= 100 k= 1 a= 1 R=INT(0.5*SS)				
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	.402672	.662145	.677672	.489239
20	.805542	.648896	.909079E-01	.0826
30	.210342	.442439	.162597	.0264
40	.673805	.454013	.013234	.015077
50	.749803E-01	.562203E-02	.495934E-02	.005951

Table 3.4.14

N= 100 k= 2 a= 2 R=INT(0.8*SS)				
SS	MB(k)	MSE(k)	MB(a)	MSE(a)
10	1.00969	1.01948	.164393	.270252
20	.14648	.214738	.448277	.200952
30	.276867	.966554E-01	.968545E-01	.198077
40	.097145	.804868E-01	.61121	.159598
50	.054138	.427897E-01	.32933	.104285

Appendix

```
1 REM     ************ PROGRAM 1 ***********************
5 REM      THIS PROGRAM IS UE TO CALCULATE THE MB AND MSE
6 REM        FOR THE LSE OF K USING CENSORED SAMPLE
10 RANDOMIZE
20 INPUT"INPUT N,A";N,A
30 PRINT "N=     ";N;"   A=   ";A;"    R=INT(3*SS/4)"
40 PRINT"*******************************************************************"
50 DIM L(1000),T(1000)
52 FOR IT=1 TO N
54 L(IT)=-LOG(1-IT/(N+1))
55 NEXT IT
60  K=1
70 PRINT"K             SS           MB               MSE"
80 PRINT"*******************************************************************"
90 FOR SS=10 TO 50 STEP 10
91 R=INT(3*SS/4)
100 S3=0
110 S4=0
120 FOR I=1 TO N
130 S1=0
140 S2=0
145 REM  READ THE SAMPLE SIZE
150 FOR J=1 TO SS
160 T(J)=((-(K+1)/A)*LOG(1-RND))^(1/(K+1))
170 NEXT J
175 REM   ARRANGE THE SAMPLE SZE
181 FOR JJ=1 TO SS
182 FOR JJJ=(JJ+1) TO SS
183 IF T(JJJ)>=T(JJ) THEN 187
184 Z=T(JJJ)
185 T(JJJ)=T(JJ)
186 T(JJ)=Z
187 NEXT JJJ
188 NEXT JJ
189 REM CALCULATE THE LSE OF K
190 Z=K
200 A1=0
210 B=0
220 C=0
230 D=0
240 E=0
250 F=0
260 G=0
270 FOR J=1 TO R
280 A1=A1+LOG(T(J))*T(J)^2*Z+2)
290 B=B+T(J)^(2*Z+2)
300 C=C+T(J)^(Z+1)*LOG(T(J))*L(J)
310 D=D+T(J)^(Z+1)*L(J)
320 E=E+(LOG(T(J)))^2*T(J)^(2*Z+2)
350 NEXT J
360 A2=A/(Z+1)*(A1*(Z+1)-B)-(C*(Z+1)-D)
370 A3=A/(Z+1)^2*((2*(Z+1)*E-A1)*(Z+1)-(A1*(Z+1)-B))-C
380 A7=A2/A3
420 Z1=Z-A7
430 IF ABS(Z1-Z)<.001 THEN GOTO 460
440 Z=Z1
450 GOTO 200
460 REM CHECK IF THE LSE OF K EXIST
465 AA=0
470 BB=0
480 CC=0
490 DD=0
500 EE=0
510 SF=0
520 FOR JJ=1 TO R
```

```
530 AA=AA+(LOG(T(JJ)))^2*T(JJ)^(Z1+1)*L(JJ)
540 BB=BB+LOG(T(JJ))*T(JJ)^(Z1+1)*L(JJ)
550 CC=CC+T(JJ)^(Z1+1)*L(JJ)
560 DD=DD+(LOG(T(JJ)))^2*T(JJ)^(2*Z1+2)
570 EE=EE+LOG(T(JJ))*T(JJ)^(2*Z1+2)
580 SF=SF+T(JJ)^(2*Z1+2)
590 NEXT JJ
600 B1=(Z1+1)*(AA*(Z1+1)^2-2*(BB*(Z1+1)-CC))
610 B2=(2*DD*(Z1+1)^2-SF-4*(EE*(Z1+1)-SF))
620 IF A>(B1/B2)THEN GO TO 650
630 PRINT "K=  ";K;" SS=  ";SS;"LSE DOESNOT EXIST"
640 GOTO 730
650 LSE=Z1
660 B=LSE-K
670 S3=S3+B
680 S4=S4+B*B
690 NEXT I
698 REM CALCULATE THE MB AND MSE
700 MB=S3/N
710 MSE=S4/N
720 PRINT K,SS,MB,MSE
730 NEXT SS
740 PRINT"**************************************************"
790 END
```

```
1 REM          ******* PROGRAM 2 ******************************
3 REM          THIS PROGRAM IS TO CALCULATE THE MB AND MSE
5 REM          FOR THE  LSE OF A AND K USING CENSORED SAMPLE
10 RANDOMIZE
20 INPUT"INPUT N K AND A";N,K,A
30 PRINT "N=    ";N;"  K=   ";K;"  A=   ";A;"   R=INT(SS/3)"
40 PRINT"************************************************************"
50 DIM L(1000),T(1000)
70 PRINT" SS      MB(K)     MSE(K)        MB(A)       MSE(A) "
80 PRINT"************************************************************"
90 FOR SS=10 TO 50 STEP 10
92 FOR Q5=1 TO SS
93 L(Q5)=-LOG(1-Q5/(SS+1))
95 NEXT Q5
100 S31=0
110 S41=0
112 S32=0
114 S42=0
120 FOR I=1 TO N
130 S1=0
140 S2=0
145 REM READ THE SAMPLE SIZE
150 FOR J=1 TO SS
160 T(J)=(-((K+1)/A)*LOG(1-RND))^(1/(K+1))
170 NEXT J
180 REM ARRANGE THE SAMPLE SIZE
181 FOR Q1=1 TO SS
182 FOR Q2=(Q1+1) TO SS
183 IF T(Q2)>= T(Q1) THEN 187
184 Z=T(Q2)
185 T(Q2)=T(Q1)
186 T(Q1)=Z
187 NEXT Q2
188 NEXT Q1
190 Z=K
195 R=INT(SS/3)
200 A1=0
210 B=0
220 C=0
230 D=0
235 F=0
240 E=0
250 REM CALCULATE THE LSE OF K
270 FOR J=1 TO R
280 A1=A1+LOG(T(J))*T(J)^(2*Z+2)
290 B=B+T(J)^(2*Z+2)
300 C=C+T(J)^(Z+1)*LOG(T(J))*L(J)
305 F=F+(LOG(T(J)))^2*T(J)^(Z+1)*L(J)
310 D=D+T(J)^(Z+1)*L(J)
320 E=E+(LOG(T(J)))^2*T(J)^(2*Z+2)
350 NEXT J
355 A2=(D*A1*(Z+1))/B-C*(Z+1)
358 A3=10*(((C*A1+2*E*D)*(Z+1)+D*A1)*B-2*(Z+1)*A1^2*D)/
       (B^2)-(F*(Z+1)+C)
360 A7=10*A2/A3
420 Z1=Z-A7
430 IF ABS(Z1-Z)<.001 THEN GOTO 460
440 Z=Z1
450 GOTO 200
455 REM CALCULATE THE LSE OF A
460 S7=0
462 S8=0
463 FOR U=1 TO R
464 S7=S7+T(U)^(Z1+1)*L(U)
465 S8=S8+T(U)^(2*Z1+2)
```

```
466 NEXT U
467 LSE1=Z1
468 LSE2=(Z1+1)*S7/S8
469 AA=0
470 BB=0
480 CC=0
490 DD=0
500 EE=0
510 SF=0
515 REM CHEK IF THE LSE OFN A AND K EXIST
520   FOR JJ=1 TO R
530  AA=AA+(LOG(T(JJ)))^2*T(JJ)^(Z1+1)*L(JJ)
540  BB=BB+LOG(T(JJ))*T(JJ)^(Z1+1)*L(JJ)
550  CC=CC+T(JJ)^(Z1+1)*L(JJ)
560  DD=DD+(LOG(T(JJ)))^2*T(JJ)^(2*Z1+2)
570  EE=EE+LOG(T(JJ))*T(JJ)^(2*Z1+2)
580  SF=SF+T(JJ)^(2*Z1+2)
590  NEXT JJ
592 LA=LSE2
593  B7=2*(DD*(Z1+1)^2)-SF
594  B1=4*(EE*(Z1+1)-SF)
595  B6=B7-B1
596  B2=(A*(Z1+1)^2-2*(B*(Z1+1)-CC))
597  B2=(2*LA/(Z1+1)^3)*B2
598  B3=(2*SF/(Z1+1)^2)*((2*LA^2/(Z1+1)^4)*B6-B2)
600  B4=(4*LA/(Z1+1)^2)*(2*(Z1+1)^2*EE-2*(Z1+1)*SF)-
       (2/(Z1+1)^2)*(BB*(Z1+1)-CC)
605  JAC=B3-B4^2
620  IF JAC>0 THEN GO TO 660
630  PRINT "K=   ";K;" SS=   ";SS;"LSE DOESNOT EXIST"
640  GOTO 730
660 B1=LSE1-K
670 S31=S31+B1
680 S41=S41+B1*B1
683 B2=LSE2-A
685 S32=S32+B2
687 S42=S42+B2*B2
690 NEXT I
695 REM CALCULATE THE MB AND MSE
700 MB1=S31/N
710 MSE1=S41/N
712 MB2=S32/N
714 MSE2=S42/N
720 PRINT USING"I ## I ##.### I ##.##### I ##.##### I
      ##.#######";SS,MB1,MSE1,MB2,MSE2
730 NEXT SS
734 PRINT"***************************************************************":E
```

```
1   REM    *********** PROGRAM 3 ******************************
5   REM      THIS PROGRAM IS USED TO CALCULATE THE MB AND
6   REM        MSE OF THE MLE FPR THE PARAMETERES OF
7   REM          THE LINEAR  HAZARD FUNCTION
8   REM            USING CENSORED SAMPLE
10 RANDOMIZE
20 DIM T(1000)
30 INPUT"INPUT n";N
35 IF (A<0 OR B<0) THEN 30
40 PRINT" N=  ";N;"  R=INT(.7*SS)"
42 A=1
44 B=2
45 PRINT"*************************************************************"
46 PRINT"  A=  ";A;"  B=";B
50 PRINT"*************************************************************"
53 PRINT" SS           MBA         MBB         MSEA          MSEB"
55 PRINT"*************************************************************"
60 FOR SS=10 TO 30 STEP 10
70 S31=0
80 S32=0
90 S41=0
100 S42=0
110 FOR I=1 TO N
111 IF B>0 THEN 120
112 FOR J=1 TO SS
113 T(J)=-LOG(1-RND)/A
114 NEXT J
115 GO TO 150
116 REM OBTAIN THE SAMPLE SIZE
120 FOR J=1 TO SS
130 T(J)=(SQRT(A^2-2*B*LOG(1-RND))/B)-(A/B)
140 NEXT J
145 REM ARRANGE THE SAMPLE SIZE
150 FOR Q1=1 TO SS
160 FOR Q2=(Q1+1) TO SS
170 IF T(Q2)>=T(Q1) THEN 210
180 Z=T(Q2)
190 T(Q2)=T(Q1)
200 T(Q1)=Z
210 NEXT Q2
220 NEXT Q1
230 R=INT(0.7*SS)
240 S1=0
250 S2=0
260 FOR J=1 TO R
270 S1=S1+T(J)
280 S2=S2+T(J)^2
290 NEXT J
300 U1=S1+(SS-R)*T(R)
310 U2=S2+(SS-R)*T(R)^2
320 G1=0
330 G2=0
340 FOR J=1 TO R
350 G1=G1+1/(2*R/U2+R/(U1*T(J)-U2/2))
360 G2=G2+(U1*T(J)-U2/2)/R
370 NEXT J
372 REM FIND THE MLES OF A AND B
380 IF G1>=0 THEN 540
390 IF G2<=0 THEN 570
400 B1=0
410 G3=0
420 G4=0
430 FOR J=1 TO R
435 KL=B1+R/(U1*T(J)-U2/2)
440 G3=G3+1/KL
```

```
445 G4=G4+((1/KL)^2)
460 NEXT J
470 B2=B1+G3/G4
480 IF ABS(B2-B1)<.001 THEN 510
490 B1=B2
500 GO TO 410
510 MLEB=B2
520 MLEA=(R-B2*U2/2)/U1
530 GO TO 590
540 MLEB=2*R/U2
550 MLEA=0
560 GO TO 590
570 MLEB=0
580 MLEA=R/U1
590 PRINT MLEB;
591 B1=MLEA-A
600 B2=MLEB-B
610 S31=S31+B1
620 S32=S32+B2
630 S41=S41+B1*B1
640 S42=S42+B2*B2
650 NEXT I
655 REM CALCULATE THE MB AND MSE
660 MBA=S31/N
670 MBB=S32/N
680 MSEA=S41/N
690 MSEB=S42/N
700 PRINT SS,MBA,MBB,MSEA,MSEB
710 NEXT SS
720 PRINT"**************************************************************************************"
740 PRINT"************************************************************************************"
760 PRINT"************************************************************************************"
```

```
1 REM     ************ PROGRAM 4 **************************
2 REM     THIS PROGRAM IS TO CALCULATE THE MB AND MSE
3 REM          FOR THE MLE OF K USING CENSORED SAMPLE
10 RANDOMIZE
20 INPUT"INPUT N,A";N,A
30 PRINT "N=     ";N;"  A=   ";A;"  R=INT(2*SS/3) "
40 PRINT"**************************************************************"
50 DIM T(1000)
60 K=2
70 PRINT"K                    SS              MB              MSE"
80 PRINT"**************************************************************"
83 FOR SS=10 TO 60 STEP 10
90 REM OBTAIN THE SAMPLE SIZE
91 FOR J=1 TO SS
92 T(J)=((-(K+1)/A)*LOG(1-RND))^(1/(K+1))
93 NEXT J
94 REM ARRANG THE SAMPLE SIZE
95 FOR Q1=1 TO SS
96 FOR Q2=(Q1+1) TO SS
97 IF T(Q2)>=T(Q1) THEN 102
98 Z=T(Q2)
99 T(Q2)=T(Q1)
100 T(Q1)=Z
102 NEXT Q2
104 NEXT Q1
109 S3=0
110 S4=0
120 FOR I=1 TO N
130 S1=0
140 S2=0
145 R=INT(2*SS/3)
150 REM FIND THE MLE OF K
190 Z=K
191 B=(SS-R)*T(R)^(Z+1)*(LOG(T(R)))^2
192 D=(SS-R)*T(R)^(Z+1)*LOG(T(R))
193 F=(SS-R)*T(R)^(Z+1)
200 A1=0
210 E=0
220 C=0
230 G=0
270 FOR J=1 TO R
280 G=G+LOG(T(J))
290 C=C+T(J)^(Z+1)*LOG(T(J))
300 E=E+T(J)^(Z+1)
310 A1=A1+T(J)^(Z+1)*(LOG(T(J)))^2
350 NEXT J
360 A2=(C+D)*(Z+1)-(E+F)
370 A3=G-(A/(Z+1)^2*A2)
380 A4=(A+B)*(Z+1)^2
390 A5=2*(((C+D)*(Z+1))-(E+F))
400 A6=(A/(Z+1)^3)*(A4-A5)
410 A7=A3/A6
420 Z1=Z+A7
430 IF ABS(Z1-Z)<.001 THEN GOTO 460
440 Z=Z1
450 GOTO 191
460 REM CHECK IF THE MLE OF K EXIST
465 AA=0
470 BB=0
520 FOR JJ=1 TO R
530 AA=AA+(LOG(T(JJ)))^2*T(JJ)^(Z1+1)
540 BB=BB+LOG(T(JJ))
590 NEXT JJ
600 B1=2*BB
610 BBB=AA+(SS-R)*T(R)^(Z1+1)*(LOG(T(R)))^2
```

```
620 IF A>(B1/BBB)THEN GO TO 650
630 PRINT K;"              ";SS;"          MLE DOESNOT EXIST"
640 GOTO 730
650 MLE=Z1
660 B=MLE-K
670 S3=S3+B
680 S4=S4+B*B
690 NEXT I
695 REM CALCULATE THE MB AND MSE
700 MB=S3/N
710 MSE=S4/N
720 PRINT K,SS,MB,MSE
730 NEXT SS
740 PRINT"********************************************************************************"
790 END
```

```
1 REM          **************** PROGRAM 5 ********************
4 REM          THIS PROGRAM IS TO CALCULATE THE MB AND MSE
5 REM          FOR THE MLE OF A AND K USING CENSORED SAMPLE
10 RANDOMIZE
20 INPUT"INPUT N  K AND A";N,K,A
30 PRINT "N=    ";N;" K=";K;"  A= ";A;"    R=INT(0.75*SS)"
40 PRINT"********************************************************************************"
50 DIM T(1000)
70 PRINT"  SS        MB(K)       MSE(K)       MB(A)        MSE(A
)"
74 PRINT"********************************************************************************"
77 FOR SS=10 TO 50 STEP 10
80 FOR J=1 TO SS
81 T(J)=((-(K+1)/A)*LOG(1-RND))^(1/(K+1))
82 NEXT J
91 FOR Q1=1 TO SS
92 FOR Q2=(Q1+1) TO SS
93 IF T(Q2)>=T(Q1) THEN 97
94 ZZZ=T(Q2)
95 T(Q2)=T(Q1)
96 T(Q1)=ZZZ
97 NEXT Q2
98 NEXT Q1
100 S31=0
110 S41=0
112 S32=0
114 S42=0
120 FOR I=1 TO N
130 S1=0
140 S2=0
145 R=INT(0.75*SS)
146 Z=K
160 D=(SS-R)*T(R)^(Z+1)*LOG(T(R))
170 F=(SS-R)*T(R)^(Z+1)
200 A1=0
210 E=0
220 C=0
230 G=0
270 FOR J=1 TO R
280 G=G+LOG(T(J))
290 C=C+T(J)^(Z+1)*LOG(T(J))
300 E=E+T(J)^(Z+1)
310 A1=A1+T(J)^(Z+1)*(LOG(T(J)))^2
350 NEXT J
357 Z1=(E+F)/((C+D)-(G/R)*(E+F))-1
430 IF ABS(Z1-Z)<.01 THEN  442
440 Z=Z1
441 GOTO 160
442 S6=0
444 MLE1=Z1
445 FOR PP=1 TO R
446 S6=S6+T(PP)^(Z1+1)
447 NEXT PP
450 MLE2=R*(Z1+1)/(S6+(SS-R)*(T(R)^(Z1+1)))
455  HH=0
460  AB=0
465  AD=0
470  FOR P=1 TO SS
475  HH=HH+(LOG(T(P)))^2*T(P)^(Z1+1)
478  AB=AB+LOG(T(P))*T(P)^(Z1+1)
480  AD=AD+T(P)^(Z1+1)
484  NEXT P
485  GG=(SS-R)*(LOG(T(R)))^2*T(R)^(Z1+1)
487  AC=(SS-R)*LOG(T(R))*T(R)^(Z1+1)
490  AE=(SS-R)*T(R)^(Z1+1)
```

```
493  FE=(HH+GG)*((Z1+1)^2)
495  FW=2*((AB+AC)*((Z1+1)^2)-(AD+AE))
496  FT=R*(FE-FW)*(Z1+1)
500  FY=((AB+AC)*(Z1+1)-(AD+AE))^2
620  IF MLE2<(FT/FY)THEN GO TO 650
630  PRINT K;"    ";A;"    ";SS;"         MLE DOESNOT EXIST"
640  GOTO 730
650  MLE1=Z1
660 B1=MLE1-K
665 B2=MLE2-A
670 S31=S31+B1
672 S32=S32+B2
680 S41=S41+B1*B1
685 S42=S42+B2*B2
690 NEXT I
700 MB1=S31/N
710 MSE1=S41/N
712 MB2=S32/N
714 MSE2=S42/N
720 PRINT SS,MB1,MSE1,MB2,MSE2
730 NEXT SS
740 PRINT"****************************************************************"
```

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VITA

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